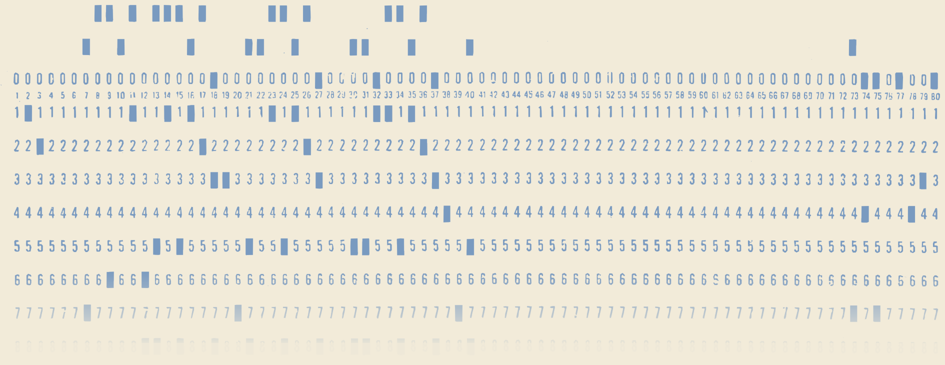

Lecture 13: Data Compression I

COMP526: Efficient Algorithms

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University of Liverpool

Updated: November 14, 2024

Announcements

1. Programming Assignment 2 posted soon
2. Quiz 5 due Friday
   - Covers string matching
   - 2 questions (multiple choice)
   - Usual rules apply
3. Attendance Code:

Meeting Goals

Discuss data compression!

- Introduce the data compression task
- Define character encoding and related terminology
- Define prefix codes
- Construct Huffman codes
- Prove optimality of Huffmann codes

Data Compression

The Story So Far

Emphasis. How do we process data?

- Data structures
  - How can we organize data perform primitive operations efficiently?
- Fundamental operations on arbitrary data:
  - sorting
  - string matching

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A New Question. How do we *store* and *transmit* data efficiently?

New Topics. Fundamental problems

1. Data Compression (starting today)
   - how to store data using as little space as possible
2. Error Correction (following topic)
   - how to

The Data Compression Task

Terminology.

- **source text:** string $S \in \Sigma_S^*$ to be stored/transmitted
  - Σ_S is some alphabet, e.g., Roman alphabet
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Goal. Represent S using as little **space** as possible.

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Our Focus: lossless compression!

The Quality of an Encoding Scheme

Goals of Encoding

- Efficiency of encoding/decoding
- resilience to errors/noise in transmission
- security (encryption)
- integrity (detect modifications)
- size

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Measure of quality. The *compression ratio:*

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- $\lceil \log \sigma \rceil$ is the minimum number of bits needed to represent σ distinct values (in binary)
- there are 2^b distinct binary strings of length b

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Interpretation. Compression ratios:

$< 1 \implies$ compression

- smaller values are better

$= 1 \implies$ no compression

$> 1 \implies$ encoded text is larger(?!)

- this is sometimes unavoidable …foreshadowing to next week

Data Compression Roadmap

Questions. When, how, and how much can we compress?

- **Part I:** Exploiting non-uniform character frequencies
  - Huffman Codes
- **Interlude:** Limits of data compression
- **Part II:** Exploiting repetition in texts
  - Run-length encoding
  - Lempel-Ziv-Welch (LZW) encoding
- **Part III:** Creating repetition in texts
  - Move-to-front transform
  - Burrows-Wheeler transform

Character Encoding

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Historical answer. ASCII use 7 bits per character

- all characters treated equally
- $2^7 = 128$ possible characters

				b7 b6 b5 →	0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
Bits b4 ↓	b3 ↓	b2 ↓	b1 ↓	Column → / Row ↓	0	1	2	3	4	5	6	7
0	0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2	STX	DC2	"	2	B	R	b	r
0	0	1	1	3	ETX	DC3	#	3	C	S	c	s
0	1	0	0	4	EOT	DC4	$	4	D	T	d	t
0	1	0	1	5	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6	ACK	SYN	&	6	F	V	f	v
0	1	1	1	7	BEL	ETB	´	7	G	W	g	w
1	0	0	0	8	BS	CAN	(	8	H	X	h	x
1	0	0	1	9	HT	EM	)	9	I	Y	i	y
1	0	1	0	10	LF	SUB	*	:	J	Z	j	z
1	0	1	1	11	VT	ESC	+	;	K	[	k	{
1	1	0	0	12	FF	FS	,	<	L	\	l	¦
1	1	0	1	13	CR	GS	—	=	M	]	m	}
1	1	1	0	14	SO	RS	.	>	N	^	n	~
1	1	1	1	15	SI	US	/	?	O	_	o	DEL

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Modern answer. Unicode

- $\sim 150{,}000$ representable characters (different scripts, emoji, etc.)
- several encoding schemes character → bits
- different characters' representations can have different lengths
    - e.g., ASCII characters represented by 8 bits

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Character Encoding. Encode each character individually $E : \Sigma_S \to \Sigma_C^*$

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- to encode a text, encode individual characters and concatenate

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Fixed vs. Variable Length Encoding

- *fixed length encoding* $\implies$ all codewords have the same length (e.g. ASCII)
- *variable length encoding* $\implies$ different lengths for different codewords (e.g. Unicode)

Fixed Length Codes

Advantages of fixed length codes

- fast decoding
  - use a lookup-table
  - can be as fast as a single array access
- local encoding
  - if character length is B, ith character starts at index $i \cdot B$

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Example. For (8-bit) ASCII encoding, how many (Roman alphabet) characters is this text? Where are the character divisions?

`011101000110010101110000 1110100`

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```
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```

Disadvantages of fixed length codes

- Inflexible (non-extensible)
  - how can we represent this awesome new emoji???
- Space inefficient
  - infrequently used characters require as much space as common characters
  - common characters are longer than they need to be

Variable Length Codes

Variable Length
Advantages:

- more flexibility
- compressibility?

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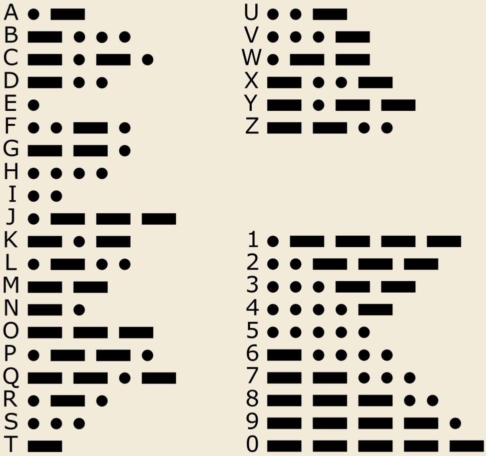
An old idea. Morse Code

- encode characters as "dots" and "dashes"
- more common characters are shorter



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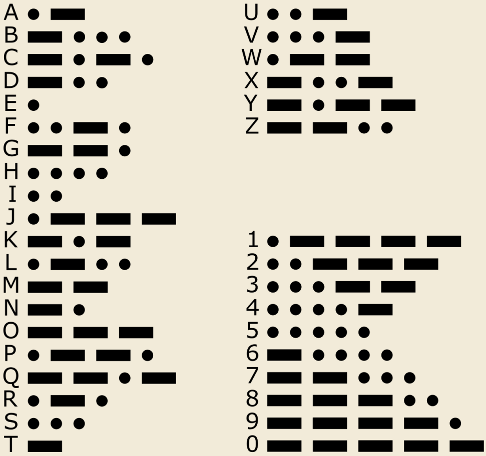
Question. How many characters in the Morse code encoding?



Codes Misbehaving

PollEverywhere

Consider the following code

c	a	n	b	s
$E(c)$	0	10	110	100

What is the original text corresponding to the encoded text 1100100100?

pollev.com/comp526

Codes Misbehaving

Question. What was the issue with this code?

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- The *relationship* between $E(n) = 10$ and $E(s) = 100$
  - If we read 10 in the encoded text, are we done reading a character?
- "Reasonable" codes should avoid this ambiguity!
  - We should *always* know when we're done reading a character.

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Prefix Codes and Tries

Definition. A character encoding E is a **prefix code** if no codeword $E(c)$ is a *prefix* of another code

Example.

c	A	E	N	O	T	␣
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Representation of prefix codes: the **trie** data structure!

- binary tree
- one leaf for each character
- edges labeled 0 or 1
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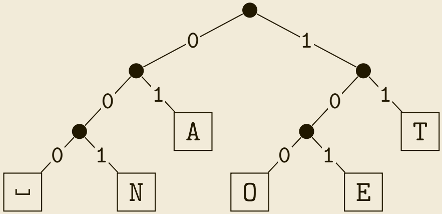

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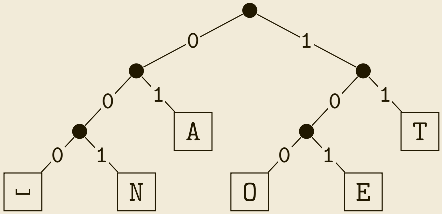

Encoding. Use the *table:* `AN␣ANT`

Decoding. Use the *trie:* `11100001010111`

Trie it Yourself

PollEverywhere Question

What is the result of using the trie on the right to decode the message:
`110010010011 1`

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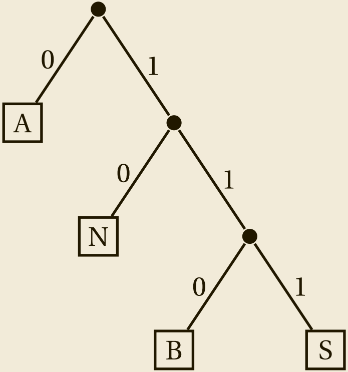

Fixed, Static, Adaptive

Note. In order to use a prefix code, we must also store the codewords!

- **fixed coding** uses the same code for all strings
  - e.g. ASCII, Unicode encodings (UTF-8)
- **static coding** uses the same codeword for each instance of a character in a text
  - codewords may different for different texts
  - must store/transmit the codewords as well as the encoded text!
- **adaptive coding** may change the codewords as the text is processed
  - codewords are stored implicitly within the coded message

Huffman Codes

Variable Length and Compression

Question. How can variable length encoding help with compression?

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Example. Consider the text `AAAAAAAAAAGGGH!`

- $\Sigma = \{$`A`,`G`,`H`,`!`$\}$
- Fixed length encoding:

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$\Longrightarrow$ Total encoded length = 30 (15 chars at 2 bits per char)

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Question. How can we find the **best possible** prefix code for compression?

Exploiting Character Frequency

Generic Optimization Problem. Suppose we are given

- a string S over the alphabet Σ;
- weights $w(c) \geq 0$ for each $c \in \Sigma$.

Find the prefix code E for Σ that minimizes $\sum_c w(c)\,|E(c)|$

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- Can we solve it *efficiently?*

Huffman Coding: Greed is Good

Idea. Build the character trie greedily from the leaves up.

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- While $|A| > 1$:
  1. u and v are two lightest vertices
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  3. set $w(p) = w(u) + w(v)$
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Example.

- $\Sigma = \{A, B, C, D, E\}$
- weights $= \{0.25, 0.15, 0.1, 0.1, 0.4\}$

LOSSLESS Example

Example. Find the Huffman encoding for the text `LOSSLESS`.

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Three Steps:

1. Compute frequency counts $w(c)$
2. Build Huffman tree
3. Write Huffman code from the tree

Huffman Analysis: Greed Works

Theorem

Given alphabet Σ and weight function $w : \Sigma \to \mathbf{R}_{\geq 0}$, the Huffman coding schemes gives the minimum weighted codeword length $\ell(E) = \sum_{c \in \Sigma} w(c) \cdot |E(c)|$ among all prefix codes.

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Proof sketch. Induction on $|\Sigma|$

- Let E^* be an optimal encoding/trie
- Claim: $\exists$ sibling leaves x, y at max depth
- Swap x and y for two min weight leaves, a, b
- Optimal code for $\Sigma' = \Sigma \setminus \{a, b\} \cup \{\boxed{\text{ab}}\}$ gives optimal code for Σ (verify this!)
- By inductive hypothesis, Huffman gives optimal code for Σ'
- So we get an optimal code for Σ $\square$

Huffman Computational Efficiency

Question. For an alphabet of size $m = |\Sigma|$ and weights w, how efficiently can we build the Huffman code?

- Maintain a collection A of active vertices
- Initially A is set of leaves, labeled with
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- While $|A| > 1$:
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- Construct the codeword table

Tie Breaking Rules

So far we have two ambiguities in our Huffman trie description:

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Conventions.

- Smaller weight child is on the left
- All ties broken by earliest character in alphabetical order
  - for internal vertices, the one containing the alphabetically first character as a descendant is on the left

Huffman and Entropy

A Thought Experiment

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- Outcome is a random point x in $[0, 1]$
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- Use binary search to find the interval!
- If the interval has width p_i need $\log(1/p_i)$ queries to determine interval
- The *expected* (average) number of queries is then

$$\mathscr{H}(p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n} p_i \log\left(\frac{1}{p_i}\right)$$

- $\mathscr{H}$ is the **entropy** of the distribution over Σ

Properties of Entropy

Setup. We choose elements from $\Sigma = \{c_1, c_2, \ldots, c_n\}$ randomly, each c_i chosen with probability p_i.

One can show:

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- If we use a Huffman encoding of Σ
  - weights $w(c_i) = p_i$
  - transmit the Huffman codeword $E(c_i)$

  Then the average length $\ell(E)$ of the transmitted word satisfies

$$\mathcal{H} \leq \ell(E) \leq \mathcal{H} + 1$$

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Conclusion. Huffman coding gives (nearly) the best possible *average* compression for *randomly* generated texts!

Emprical Entropy

Definitions. For a fixed string S over alphabet $\Sigma = \{c_1, c_2, \ldots, c_\sigma\}$, we define the **relative frequency** of character c_i in S to be

$$p_i = \frac{\#\text{ occurrances of } c_i \text{ in } S}{|S|}$$

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The length of the Huffman encoded text $C = E(S)$ is

$$|C| = \sum_{i=1}^{\sigma} |S|_{a_i} |E(c_i)| = n \sum_{i=1}^{n} p_i |E(c_i)| = n\ell(E).$$

Applying the previous slide gives $\mathcal{H}_0(S)\, n \le |C| \le (\mathcal{H}_0(S) + 1)\, n$.

- Entropy and Huffman coding length are intimately connected

Next Time

More Compression!

- Limits of Compressibility
- Compressing Repetitive Texts

Scratch Notes