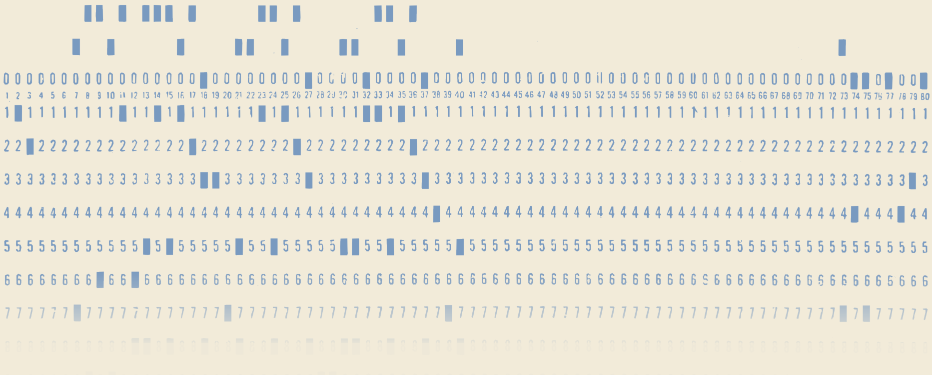

Lecture 12: String Matching III

COMP526: Efficient Algorithms

Updated: November 12, 2024

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University of Liverpool

Announcements

1. Programming Assignment 1 **DUE WEDNESDAY**
    - Use updated testing code (from last Wednesday)
    - Submission through Canvas
    - *Only* submit `pr_tester.py`
    - Late Policy: 5% off per day down to 50%
2. Quiz due Friday
    - Covers string matching
        - *including today's lecture*
    - 2 questions (multiple choice)
3. Attendance Code:

Meeting Goals

Discuss String Matching procedures:

- Knuth-Morris-Pratt
- Boyer-Moore

The String Matching Problem

Input:

- A **text** $T \in \Sigma^*$ of length n
- A **pattern** $P \in \Sigma^*$ of length m

Output:

- The index of the **first occurrence** of P in T

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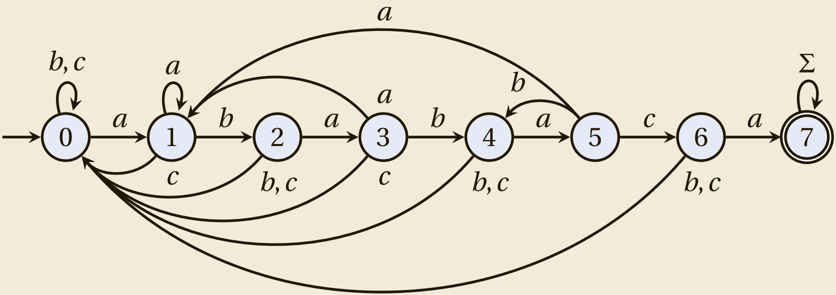
Last Time. Search with DFA



Example: $T =$ `abababac`

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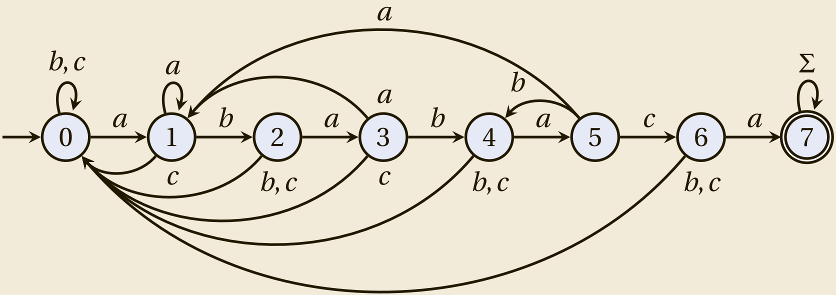

Result: Search in time $\Theta(n + |\Sigma|\, n)$ with space overhead $|\Sigma|\, n$.

Knuth-Morris-Pratt

Failure Link Automaton

DFA efficiency.

- Space/time to build DFA: $\Theta(m|\Sigma|)$
- Time to execute DFA: $\Theta(n)$

$\Longrightarrow$ Overall time is $\Theta(n + m|\Sigma|)$

  - additional space overhead is $\Theta(m|\Sigma|)$

Question. Can we perform string matching in time $O(n)$ with *less space overhead?*

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Question. Can we perform string matching in time $O(n)$ with *less space overhead?*

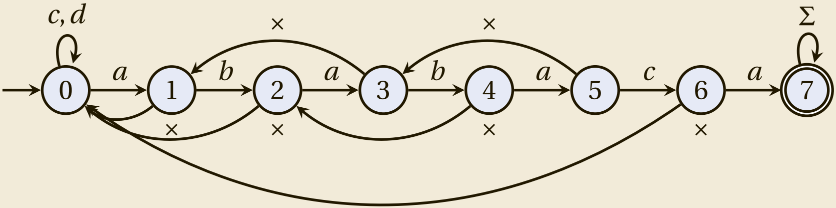
Idea. When comparison fails, don't have a separate transition for each failing character

- Just record failure and "shift" pattern as far forward as possible

Failure Link Automaton

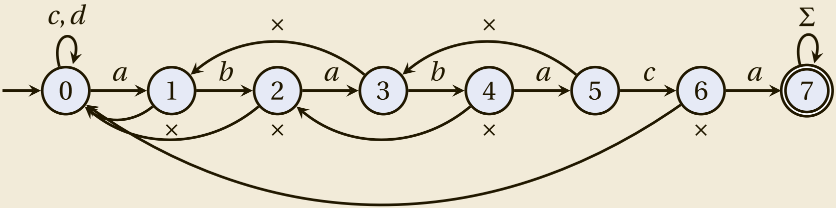
Example

- T = aababaababacaa
- P = ababaca



text	*a*	*a*	*b*	*a*	*b*	*a*	*a*	*b*	*a*	*b*	*a*	*c*	*a*	*a*
states														

States and Shifts



a	*a*	*b*	*a*	*b*	*a*	*a*	*b*	*a*	*b*	*a*	*c*	*a*	*a*
a	*b*	*a*	*b*	*a*	*c*	*a*							
	a	*b*	*a*	*b*	*a*	*c*	*a*						
			a	*b*	*a*	*b*	*a*	*c*	*a*				
					a	*b*	*a*	*b*	*a*	*c*	*a*		
						a	*b*	*a*	*b*	*a*	*c*	*a*	

Correspondence: matches increment T index i, mismatches shift P

- shift amount aligns largest possible number of matches

FLA Execution

A **Failure Link Automaton** (**FLA**) consists of:

- A finite set Q of **states**
- A finite **alphabet** Σ
- A **transition function** $\varphi : Q \times (\Sigma \cup \{\times\}) \to Q$
- An **initial state** $q_0 \in Q$
- A set $F \subseteq Q$ of **accepting states**

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Execution. To apply and FLA to T

- Start at the state q_0
- Read characters from T sequentially
  - if in state q and read character c:
    - if $\varphi(q, c)$ is defined, move to state $\varphi(q, c)$
    - otherwise move to state $\varphi(q, \times)$ and **re-read** c
- Return TRUE if end in "accepting" state

FLA Execution

PollEverywhere Question

Given an FLA for a pattern P of length m, how many times could we follow failure links for a single character c read from T in the worst case?

pollev.com/comp526

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- Start at the state q_0
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FLA Running Time

More careful analysis

- If we match up to $P[j]$, then we can only follow up to j back links
- In order to witness j failures, must have witnessed j successes!

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Amortized cost of each character read from T

- If read character c is a **match**:
  - pay 1 for comparison
  - put 1 unit cost in the **bank**
- If read character c is a **mismatch**
  - *withdraw* 1 from the bank
- By analysis above account balance is always non-negative

$\Longrightarrow$ amortized cost of each comparison is 2

$\Longrightarrow$ hence overall running time of execution is $O(n)$

FLA Construction

Observation. Each state q has

- 1 forward link to state $q+1$
- 1 fail link

Given P, we don't need to store forward link label:

- forward link label from q to $q+1$ is $P[q]$

Only need to store fail link state!

- this can be stored as a single array of size m

$\Longrightarrow$ only $O(m)$ space overhead

FLA Construction

Definition. The **failure link array** *fail* of P the array of m numbers that stores the (index of) the next state for each failure

- How do we construct it?

FLA Construction

Definition. The **failure link array** *fail* of P the array of m numbers that stores the (index of) the next state for each failure

- How do we construct it?
- Again x is length of largest prefix that matches a suffix of $P[1, q)$

Example. $P[0..6) =$ `ababaca`

q	0	1	2	3	4	5	6
$fail[q]$							

```
procedure FAILURELINK(P[0, m))
    fail[0] ← 0
    x ← 0
    for j = 1, 2, ..., m − 1 do
        fail[j] ← x
        while P[x] ≠ P[j] do
            if x = 0 then
                x ← −1
                break
            else
                x ← fail[x]
            end if
        end while
        x ← x + 1
    end for
end procedure
```

FLA Construction

Question. What is the running time of FAILURELINK on input of size m?

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FLA Construction

Question. What is the running time of FAILURELINK on input of size m?

Observations.

- x incremented once per j
- $fail[x] < x$
- Each “while” iteration decrements x

So at most $2m$ updates to x

- cf. amortized analysis
- x = bank balance

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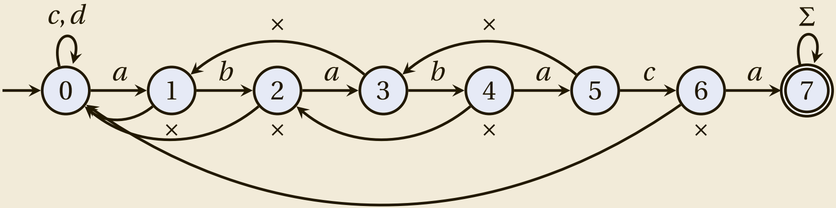
Failue Links: 3 Views



$a \quad b \quad a \quad b \quad a \quad c \quad a$

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$\mathit{fail}[q]$	0	0	0	1	2	3	0

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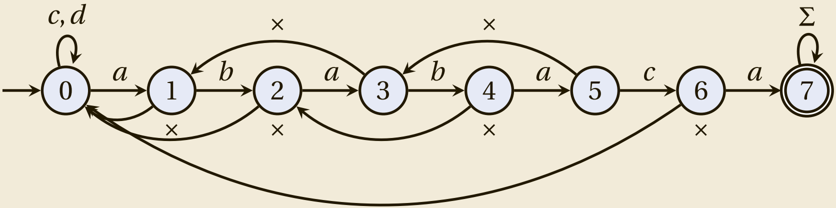

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q	0	1	2	3	4	5	6
fail$[q]$	0	0	0	1	2	3	0

fail$[q]$ is

- the max of alignments formed by shifting P if first mismatch at $P[q]$
- longest prefix of $P[0,q)$ that is a suffix of $P[1,q)$

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- Scan along $T[0, n)$
  - index i
- Maintain position in $P[0, m)$
  - index j
  - current prefix match
- When $T[i] = P[j]$, increment i and j
- Otherwise, $j \leftarrow \mathit{fail}[j]$
  - unless $j = 0$, then $i \leftarrow i + 1$

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```
procedure KMP(T[0..n), P[0..m))
    fail ← FAILURELINK(P)
    i ← 0
    j ← 0
    while i < n do
        if T[i] = P[q] then
            i ← i + 1, j ← j + 1
            if j = m then return i − j
        else
            if j ≥ 1 then
                j ← fail[j]
            else
                i ← i + 1
            end if
        end if
    end while
end procedure
```

KMP Algorithm

Analysis:

- Running time $O(n+m)$
  - $O(m)$ to build *fail*
  - $O(n)$ to apply KMP
  - analysis uses **amortized analysis**
- Additional space $O(m)$
  - just need to store *fail* and indices

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Clean Takeaway:

fail$[j]$ is the length of the longest prefix of $P[0..j]$ that is a suffix of $P[1..j]$

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KMP Example

Example. Find the failure link array for $P[0,8) = \texttt{BCBABCBA}$.

i	0	1	2	3	4	5	6	7
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Interpretation. If $T[i..i+j)$ matches $P[0..j)$, but $T[i+j] \neq P[j]$, then $fail[j]$ is the maximum number matches between $T[i+1,i+j]$ and P.

1	2	3	4	5	6	7					
B	C	B	A	B	C	B	A				
		B	C	B	A	B	C	B	A		
				B	C	B	A	B	C	B	A

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				B	C	B	A	B	C	B	A

Visualization. See website.

DFA vs FLA

Question. Which is better? DFA matching or KMP algorithm?

- KMP has overall running time $O(n + m)$
  - amortized 2 comparisons per T access
- DFA has overall running time $O(n + m|\Sigma|)$
  - 1 comparison per T access
  - $|\Sigma|$ dependence

Boyer-Moore

Beyond Worst-Case Pattern Matching?

A Puzzle. Suppose we have

- $P[0,4) = \texttt{AAAA}$
- $T[0,14) = \texttt{BBBBBBBBBBBBBB}$

If we know P, what is the fewest number of accesses we can make to T to **certify** that T does not contain P?

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?	?	?	B	?	?	?	B	?	?	?	B	?	?
A	A	A	A										
				A	A	A	A						
								A	A	A	A		

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A	A	A	A										
				A	A	A	A						
								A	A	A	A		

Observation.

- By starting comparisons from the *end* of P, we could eliminate more possible alignments.

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Strategy. To test match of $P[0..m)$ with $T[j..j+m)$, perform comparisons from *right to left*

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Heuristic 1. If we encounter $T[i]$ that does not occur in P, shift P entirely past index i.

T:	$\cdots$	A	B	D	C	A	A	C	A	B	C	A	$\cdots$
P:		C	A	B	C	A							
				$\rightarrow$	C	A	B	C	A				

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P:		C	A	B	C	A							
				→	C	A	B	C	A				

Heuristic 2. If we match on a suffix of P but mismatch at index i, shift P to next alignment of suffix.

T:	⋯	A	B	D	C	A	A	C	A	B	C	A	⋯
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Boyer-Moore Algorithm

Combining these heuristics gives the **Boyer-Moore algorithm**

- Compare alignments from right to left
- If we encounter $T[i]$ that does not occur in P, shift P entirely past index i.
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Features of this approach:

- Worst-case running time on $P[0..m)$ and $T[0..n)$ is $\Theta(nm)$
  - achieved if all instances of P must be reported
  - can be improved to $\Theta(n+m+|\Sigma|)$ with some care if T does not contain P

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  - For some *random* string models, expected running time is $O(n/m)$
  - For English text, typically uses $\sim 0.25n$ comparisons if no match

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- Space overhead is $\Theta(m + |\Sigma|)$

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  - $\Theta(nm)$ worst case to find all occurrences, can be as small as $O(n/m)$
  - $\Theta(m)$ overhead
- **Rabin-Karp**
  - based on **hashing**
  - generalizes beyond one-dimensional strings
  - expected running time $O(n+m)$
  - $O(1)$ space overhead

Next Time

Data Compression!

- How much **space** do we need to store our data?

Scratch Notes