Lecture 4: Data Structures I

COMP526: Efficient Algorithms

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Updated: October 15, 2024

Announcements

1. Second Quiz, due Friday
   - Similar format to before
     - One question, select all correct answers
     - 20 minute time limit
   - Covers asymptotic (Big-O) notation
     - Lectures 03 and 04
     - Relevant reading from **CLRS** — Intro to Algorithms
   - Quiz is **closed resource**
     - No books, notes, internet, etc.
     - Do not discuss until after submission deadline (Friday night, after midnight)
2. Programming Assignment 1: Discuss on Thursday
   - Due 13 November
3. Attendance Code: 239179

Meeting Goals

- Finish discussion of asymptotic notation
- Introduce Abstract Data Types:
  - Stack
  - Queue
  - Priority Queue
- Discuss array-backed and linked list-backed implementations of Stacks and Queues
- Introduce amortized analysis

Asymptotic Notation

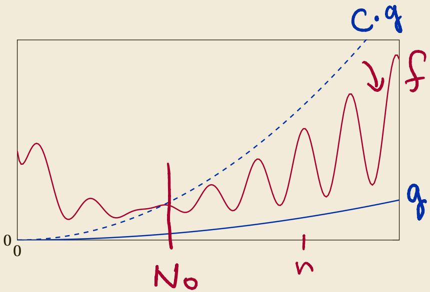
From Last Time

Definition

Suppose f and g are functions from $\mathbf{N}$ to $\mathbf{R}^+$. Then we say that $f = O(g)$ (read: f is *big O of g*) if there exist constants $N_0 \in \mathbf{N}$ and $C \in \mathbf{R}$ such that for all $n \in \mathbf{N}$

$$n \geq N_0 \implies f(n) \leq Cg(n).$$

Equivalently, $f = O(g) \iff \limsup \frac{f(n)}{g(n)} < \infty$

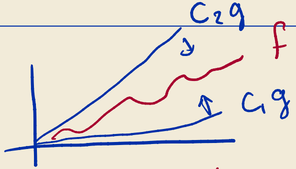


Proposition

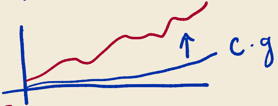
Suppose $f, f_1, f_2, g, g_1, g_2, h$ are functions and a is any constant. Then:

1. $(\forall n\, f(n) \leq a) \implies f = O(1)$
2. $(\forall n\, f(n) \leq g(n)) \implies f = O(g)$
3. $f = O(g) \implies a \cdot f = O(g)$
4. $f = O(g)$ and $g = O(h) \implies f = O(h)$
5. $f = O(h)$ and $g = O(h) \implies f + g = O(h)$
6. $f_1 = O(g_1)$ and $f_2 = O(g_2) \implies f_1 \cdot f_2 = O(g_1 \cdot g_2)$

Variations of O



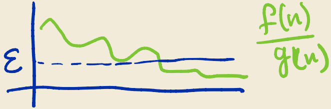
- $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$
  - Example: $4n^2 + 3n + 7 = \Theta(n^2)$



- $f = \Omega(g)$ if $g = O(f)$
  - Example: $0.01n^2 - 7n = \Omega(n^2)$

Informally: f much smaller than g

- $f = o(g)$ if for every $\varepsilon > 0$, there exists N_0 such that $n \geq N_0 \implies \frac{f(n)}{g(n)} < \varepsilon$.
  - Equivalently:
    $f = o(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
  - Example: $n^{1.999} = o(n^2)$



Informally f grows significantly faster than g

- $f = \omega(g)$ if $g = o(f)$
  - Example: $0.01n^{2.01} = \omega(n^2)$

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Mnemonic for Variations

Big-O	(in)equality
ω	$>$
Ω	$\geq$
Θ	$\approx$
O	$\leq$
o	$<$

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More Properties

- $f_1 = O(g_1)$ and $f_2 = o(g_2) \implies f_1 \cdot f_2 = o(g_1 \cdot g_2)$
- $f_1 = \Omega(g_1)$ and $f_2 = \omega(g_2) \implies f_1 \cdot f_2 = \omega(g_1 \cdot g_2)$

Interpretation

Suppose:

- two algorithms A and B for solving the same problem
- running time of A is f, running time of B is g
- $f = o(g)$ — f grows strictly slower than g

Consider running A on a slow machine M_1 and B on a fast machine M_2. Then: regardless of how much slower M_1 is than M_2, for *sufficiently large* inputs, A will complete faster than B.

Interpretation

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- two algorithms A and B for solving the same problem
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The Moral. Efficient *algorithms* are better than faster hardware.

- little-o notation gives the “right” abstraction to formalize this relationship

Common Orders of Growth

Named orders of growth:

name	asymptotic growth
constant	$O(1)$
logarithmic	$O(\log n)$
polylogarithmic	$O(\log^c n)$
linear	$O(n)$
almost linear	$O(n \log^c n)$
quadratic	$O(n^2)$
polynomial	$O(n^c)$
exponential	$O(c^n)$

base of log doesn't matter for big O notation:

$$\log_b a = \frac{\log_2 a}{\log_2 b}$$

b: const; $\log_2 b$: const.

$= (\log n)^c$

c = any constant value

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increasing rates of growth

Relationships

Between classes: constant

For all $a, b > 0$

- $a = o(\log^b n)$
- $\log^a n = o(n^b)$
- $n^a = o(b^n)$

Common Orders of Growth

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If $f = o(g)$ then $f = O(g)$

$n! = n \times (n-1) \times (n-2) \cdots 1 < n^n$

Relationships

Between classes:
For all $a, b > 0$

- $a = o(\log^b n)$
- $\log^a n = o(n^b)$
- $n^a = o(b^n)$

Within classes:
For all a, b, $a < b$

- $\log^a n = o(\log^b n)$
- $n^a = o(n^b)$
- $a^n = o(b^n)$

faster than exp.

$n \cdot n-1 \cdots \frac{n}{2} > \left(\frac{n}{2}\right)^{\frac{n}{2}}$

Example

Example

Compare the asymptotic growth of the following functions:

1. $f(n) = 2n^2 + 2^{n/2}$
2. $g(n) = \log^2 n + \sqrt{n}$
3. $h(n) = n + n\log n + n^{3/2}$

polynomial

Exponential

a^n

$\frac{1}{2} \cdot 2^n$

$2^{n/2} = 2^{\frac{1}{2} \cdot n} = (2^{1/2})^n$

$2^{1/2} = 1.4\ldots = c$

c^n for $c = \frac{1}{2}$

$\sqrt{n} = n^{1/2}$

$O(n^{3/2})$

$g(n) = \Theta(n^{1/2})$

$\Theta(c^n)$

$o(c^n) + \ldots = \Theta(c^n)$

Also $\Omega(c^n)$

$\Downarrow$

$\Theta(c^n)$

$n \log n \;?\; n \cdot n^{1/2}$

$\log n = o(n^{1/2})$

$\Downarrow$

$n \log n = o(n^{3/2})$

$h(n) = \Theta(n^{3/2})$

$(a^b)^c = a^{b \cdot c}$

$f = \omega(g)$

$g = o(h)$

Linear ADTs and Data Structures

Abstract Data Types and Data Structures

Abstract Data Types (ADTs)

An **abstract data type** gives a formal specification of a task to be performed:

many operations

- List of supported operations (syntax)
- The effects of applying the operations (semantics)

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Specify WHAT

Data Structures

A **data structure** specifies

- how data is represented
- how the supported operations are performed (i.e., what algorithms are used)
- what are the costs of the operations

Specify HOW

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- what are the costs of the operations

Question. Why is it useful to separate ADTs from Data Structure?

- Can swap different data structures for same ADT
  - applications *using* the functionality will not be broken
  - different data structures may be more efficient in some applications
- Better abstractions
- Generic lower bounds

The Stack ADT

Stacks, Intuitively

Goal: to store a *collection* of elements

- elements arranged as in a stack of books
- can only access top-most element:
  - put a new book on the stack
  - look at the top-most book
  - remove the top-most book

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Stacks, Formally

- S is the state of the stack, initially $S = \varnothing$
- $S.\textsc{Push}(x) : S \mapsto Sx$ (concatenation)
- $S.\textsc{Top}() :$ returns x_{n-1} where $S = x_0 x_1 \cdots x_{n-1}$ (last elt. in S)
- $S.\textsc{Pop}() : Sx \mapsto S$, returns x
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Tons of Applications!

- Executing programs (call stack)
- Parsing/evaluating arithmetic expression
- Syntax checking (parenthesis)
- ...

Try It Yourself!

PollEverywhere Question

What is the result of calling TOP() after the following sequence stack operations:

PUSH(1)
PUSH(2)
PUSH(3)
POP()
PUSH(4)
PUSH(5)
POP()
PUSH(6)
POP()
POP()

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Try It Yourself!

PollEverywhere Question

What is the result of calling Top() after the following sequence stack operations:

Push(1)

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Push(3)

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Pop()

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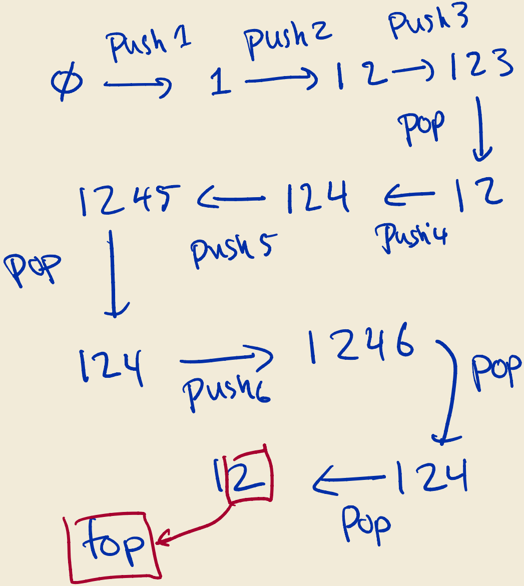
Pop()



Linked List Backed Stack Implementation

Idea

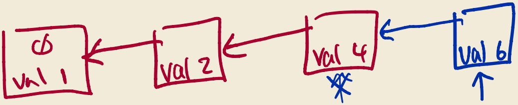
- Store each element in a NODE
- Each NODE stores
  - the value of an element in the stack
  - a *reference* to the NODE storing the next element

```
class NODE
    datavalue
    NODE next
end class
```

points to node "below" in stack

ignores empty stack condition

```
class LISTSTACK
    NODE head
    procedure PUSH(x)
        n ← new NODE
        n.data ← x
        n.next ← head
        head ← n
    end procedure
    procedure POP
        n ← head
        head ← n.next
        return n.data
    end procedure
    procedure TOP
        return head.data
    end procedure
end class
```



State S = 1 2 4

push(6)

Issues with Linked List Stacks

Issues

- NODEs waste space
  - must store reference for each entry

Memory bad

- Following chains of reference is costly
  - memory access is non-local
  - sequential memory access is more efficient

```
class LISTSTACK
    NODE head ← ∅
    procedure PUSH(x)
        n ← new NODE
        n.data ← x
        n.next ← head
        head ← n
    end procedure
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Arrays as ADTs

Informally, arrays are indexed lists of elements:

$a =$

0	1	2	3	4	5	6	7	8
l	i	v	e	r	p	o	o	l

Array Operations (ADT):

- **create** an array of size n
- **get** the element at index i:
  - $a[4]$ returns r
- **set** the value at index i to a prescribed value
  - $a[5] \leftarrow c$

Array Operation Costs (Data Structure)

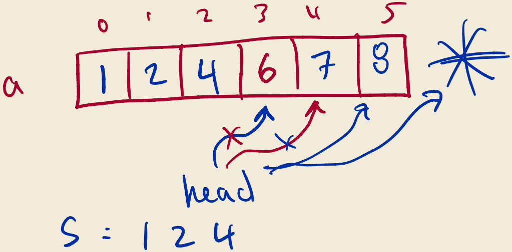
- **create** an array of size n has cost $O(n)$
- **get** and **set** have cost $O(1)$ ← arrays are great!

Array Backed Stack Implementation

Idea:

- Store elements in the stack in an array
  - access array values by *index*
  - neighboring values at adjacent indices

    $\Longrightarrow$ sequential access
- Only overhead: store index of head (top)




```
class ArrayStack
    a ← new array
    head ← 0
    procedure Push(x)
        a[head] ← x
        head ← head + 1
    end procedure
    procedure Pop
        head ← head − 1
        return a[head]
    end procedure
    procedure Top
        return a[head − 1]
    end procedure
end class
```

Push(6)
Push(7)

Push(8)
Push(9)

Array Backed Stack Implementation

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What is the issue here?

Arrays have fixed size.

Resizing Arrays

The Problem: Arrays are *fixed size!*

- What if we don't know the (maximum) size of the stack in advance?

Resizing Arrays

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- What if we don't know the (maximum) size of the stack in advance?

A Solution: Make a larger array when necessary!

- Must copy contents of old array into new array…
  …this is costly!

Increasing stack capacity

```
class ArrayStack
    a ← new array
    ...
    procedure IncreaseCapacity(k)
        n ← Size(a)
        b ← new array of size n + k        (additional space)
        for i = 0, 1, ..., n − 1 do
            b[i] ← a[i]
        end for
        head ← ~~k~~ n in b
    end procedure
end class
```

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        for i = 0, 1, …, n − 1 do
            b[i] ← a[i]
        end for
        head ← b
    end procedure
end class
```

$O(1)$ $O(n+k)$ $O(n)$ $O(1)$

Question. What is the running time of IncreaseCapacity?

$O(n+k)$

Two Strategies

Design Question. When our array runs out of room, by how much should we increase the stack capacity?

Strategy 1. Increase the capacity by $k = 1$ each time.

- Why increase the size more than we need to?

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Strategy 2. Increase the capacity by n each time!

- Maybe we'll need more extra space?

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PollEverywhere Question

Which strategy will lead to better performance?

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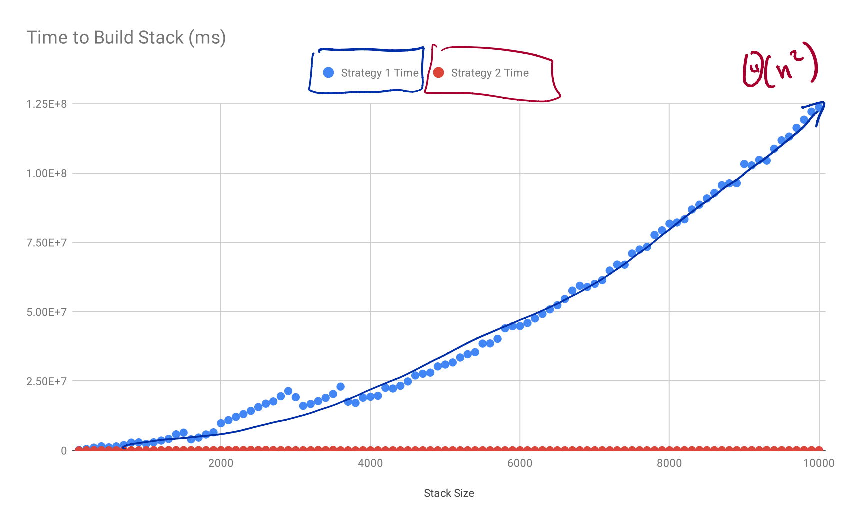
Running Time Comparison



Understanding the Discrepancy

Question. Why was the difference in running time so dramatic?

Observation. Both strategies have *worst-case* running time of $\Theta(n)$ for INCREASECAPACITY

- Strategy 1 may incur this on *every* PUSH operation
  - Overall running time $\Theta(n^2)$

Understanding the Discrepancy

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- Strategy 1 may incur this on *every* PUSH operation
  - Overall running time $\Theta(n^2)$
- For Strategy 2, INCREASECAPACITY only gets called when the stack size is $1, 2, 4, 8, \ldots, 2^k, \ldots, n$.
  - If cost of resizing n' is $c \cdot n'$, what is total resize cost?

$c \cdot 1 + c \cdot 2 + c \cdot 4 + c \cdot 8 + \cdots cn$

$c \cdot 1 + c \cdot 2 \cdots c\frac{n}{4} + c\frac{n}{2} + cn$

$cn\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) \leq 2$

$2 \cdot cn = \Theta(n)$

On average each push is $O(1)$

Amortized Analysis

Goal. To analyze the worst-case running time of a *sequence* of operations.

- Amortized cost = largest average cost per operation averaged over all sequences.

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Banker's View

- Each operation has a (financial) cost
- Cost can be paid:
  - from pocket
  - from bank account
- For each operation, can
  - withdraw from account
  - deposit to account

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A sequence of operations has amortized cost *c* if for each operation:

1. the operation is paid for (from pocket or bank account)
2. at most *c* value is paid from pocket and/or *deposited* during each operation

Example $c = 10$
- op cost 2
⇒ pay 2 from c for op
deposit $c - 2 = 8$ to bank

$B = 100$
- next op cost 60
- pay 10 out of pocket and 50 from bank → new bal = 50

Amortized Analysis of Strategy 2

Setup. Suppose we apply Strategy 2 (double the capacity when full):

- PUSH(x) has cost $c_1 = O(1)$ if the array is not full,
- PUSH(x) has cost $c_2 = O(n)$ if the array is full.

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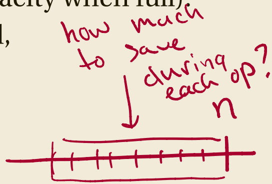

PollEverywhere Question

How much money must we add to our bank account after each (not full) PUSH to ensure our balance is at least c_2 before the next resize?

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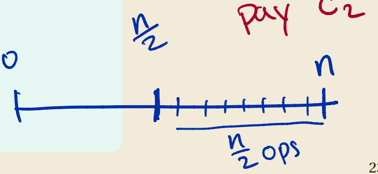

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Completing the analysis:

- If current capacity is n, last resize was at capacity $n/2$ ← because double the size
- There were (at least) $n/2$ non-resizing PUSH operations before next resize
- Must pay c_2 for next resize
- It suffices to put $c_2/(n/2) = 2c_2/n$ in bank each operation

On each non-resizing operation, we pay c_1 out of pocket, and $2c_2/n$ into the bank

$\implies$ the amortized cost is $c_1 + 2c_2/n = O(1) + \frac{1}{n}O(n) = O(1)$.

The Moral. A single resize may cost $\Theta(n)$, but the average cost over sequences of operations is always $O(1)$ (if we're careful).

The Queue ADT

Queues, Intuitively

Goal: to store a *collection* of elements

- elements arranged as in a queue at Tesco
- new people enter the **back** of the queue
- only the person at the **front** of the queue can be removed (serviced)

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Queues, Formally

- S is the state of the queue, initially $S = \varnothing$
- $S.\textsc{Enqueue}(x) : S \mapsto xS$
- $S.\textsc{Front}()$: returns x_{n-1} where $S = x_0 x_1 \cdots x_{n-1}$
- $S.\textsc{Dequeue}() : Sx \mapsto S$, returns x
- $S.\textsc{Empty}()$ returns $\textsc{True} \iff S = \varnothing$

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Tons of Applications!

- Scheduling
- Messaging
- …

Queues, Formally

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List Backed Queues

Idea

- Store each element in a NODE
- Store references to NODE:
  - head at the front of the queue
  - tail at the back of the queue

```
class ListQueue
    Node head
    Node tail
    procedure Enqueue(x)
        n ← new Node
        n.data ← x
        tail.next ← n
        tail ← n
    end procedure
    procedure Dequeue
        n ← head
        head ← n.next
        return n.data
    end procedure
end class
```

List Backed Queues

Idea

- Store each element in a NODE
- Store references to NODE:
  - head at the front of the queue
  - tail at the back of the queue

Issues:

- Similar to linked list stack implementation
  - Locality of reference
  - NODE memory overhead

```
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    Node head
    Node tail
    procedure Enqueue(x)
        n ← new Node
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```

Array Backed Queues

Idea:

- Store elements in the stack in an array
- Maintain indices of head and tail

Ignores resizing/checking if full

```
class ArrayQueue
    a ← new array, size n
    head, tail ← 0
    procedure Enqueue(x)
        a[tail] ← x
        tail ← tail + 1
    end procedure
    procedure Dequque
        head ← head + 1
        return a[head − 1]
    end procedure
end class
```

Array Backed Queues

Idea:

- Store elements in the stack in an array
- Maintain indices of head and tail

What is the problem here?

Ignores resizing/checking if full

```
class ArrayQueue
    a ← new array, size n
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    procedure Enqueue(x)
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```

Array Backed Queues

Idea:

- Store elements in the stack in an array
- Maintain indices of head and tail

The fix:

- Use *circular arrays*
- Perform index arithmetic *modulo n* (array size)
- All operations are then $O(1)$
  - amortized $O(1)$ time if resizing by doubling size

Ignores resizing/checking if full

```
class ArrayQueue
    a ← new array, size n
    head, tail ← 0
    procedure Enqueue(x)
        a[tail] ← x
        tail ← tail + 1 mod n
    end procedure
    procedure Dequque
        head ← head + 1 mod n
        return a[head − 1 mod n]
    end procedure
end class
```

The (Min) Priority Queue ADT

Priority Queues, Intuitively

Goal: to store a *collection* of elements

- Each element x has an associated *priority*, $p(x)$
- New elements **inserted** with prescribed priorities
- Can access/remove element with the *minimum* priority in the collection

The (Min) Priority Queue ADT

Priority Queues, Intuitively

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Priority Queues, Formally

- S is the state of the queue, initially $S = \varnothing$
- $S.\textsc{Insert}(x, p(x)) : S \mapsto xS$
- $S.\textsc{Min}() :$ returns x_0 where $S = x_0 x_1 \cdots x_{n-1}$
- $S.\textsc{RemoveMin}() : xS \mapsto S$, returns x
- $S.\textsc{DecreaseKey}(x, p')$ $S = x_0 x_1 \cdots x_{i-1} x x_{i+1} \cdots x_{n-1} \mapsto x_0 x_1 \cdots x_{j-1} x x_j x_{i-1} x_{i+1} \cdots x_{n-1}$
    - $p(x_j) \leq p'(x) < p(x_{j+1})$

The (Min) Priority Queue ADT

Priority Queues, Intuitively

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  - $p(x_j) \le p'(x) < p(x_{j+1})$

For Next Time

- Think about implementing min priority queues with linked lists and stacks
- Consider the running times of the priority queue operations

Next Time: Trees!

- Heaps
- Binary Search Trees
- Balanced Binary Trees

Scratch Notes