Lecture 11: Solving Recurrences

COSC 311 *Algorithms*, Fall 2022

Announcements

Midterm 10/07

- Yes, there will be a makeup exam the following week
- Midterm guide posted next weekend

Overview

1. Finishing Multiplication
2. Solving Recurrences
3. Maximizing Profit

Last Time

1. Gradeschool multiplication in binary
2. Less intuitive multiplication

a, b n-bits

$a * b$ in time $O(n^2)$

Multiplication via Divide and Conquer

Idea. Break numbers up into parts

- Assume a and b are both represented with $n = 2B$ bits, n power of 2
- Write:
  - $a = a_1 a_0 = a_1 2^B + a_0$
  - $b = b_1 b_0 = b_1 2^B + b_0$

$a = \boxed{a_1} \boxed{a_0}$

$b = \boxed{b_1} \boxed{b_0}$

$a_1 \underbrace{0 \cdots 0}_{B} + \underbrace{00 \cdots 0}_{B} a_0 = a_1 a_0 = a$

- Then:

$$\begin{aligned} ab &= (a_1 2^B + a_0)(b_1 2^B + b_0) \\ &= a_1 b_1 2^{2B} + (a_1 b_0 + a_0 b_1) 2^B + a_0 b_0 \end{aligned}$$

The Trick

- With $ab = a_1 b_1 2^{2B} + (a_1 b_0 + a_0 b_1) 2^B + a_0 b_0$
- Compute:
  - $c_2 = a_1 b_1$
  - $c^* = (a_1 + a_0)(b_1 + b_0)$
  - $c_0 = a_0 b_0$
- Then:

$$ab = c_2 2^{2B} + (c^* - c_2 - c_0) 2^B + c_0$$

Conclusion. Each a multiplication of size n can be computed using 3 multiplications of size $n/2$ and $O(1)$ addition/subtractions/shifts.

Karatsuba Multiplication

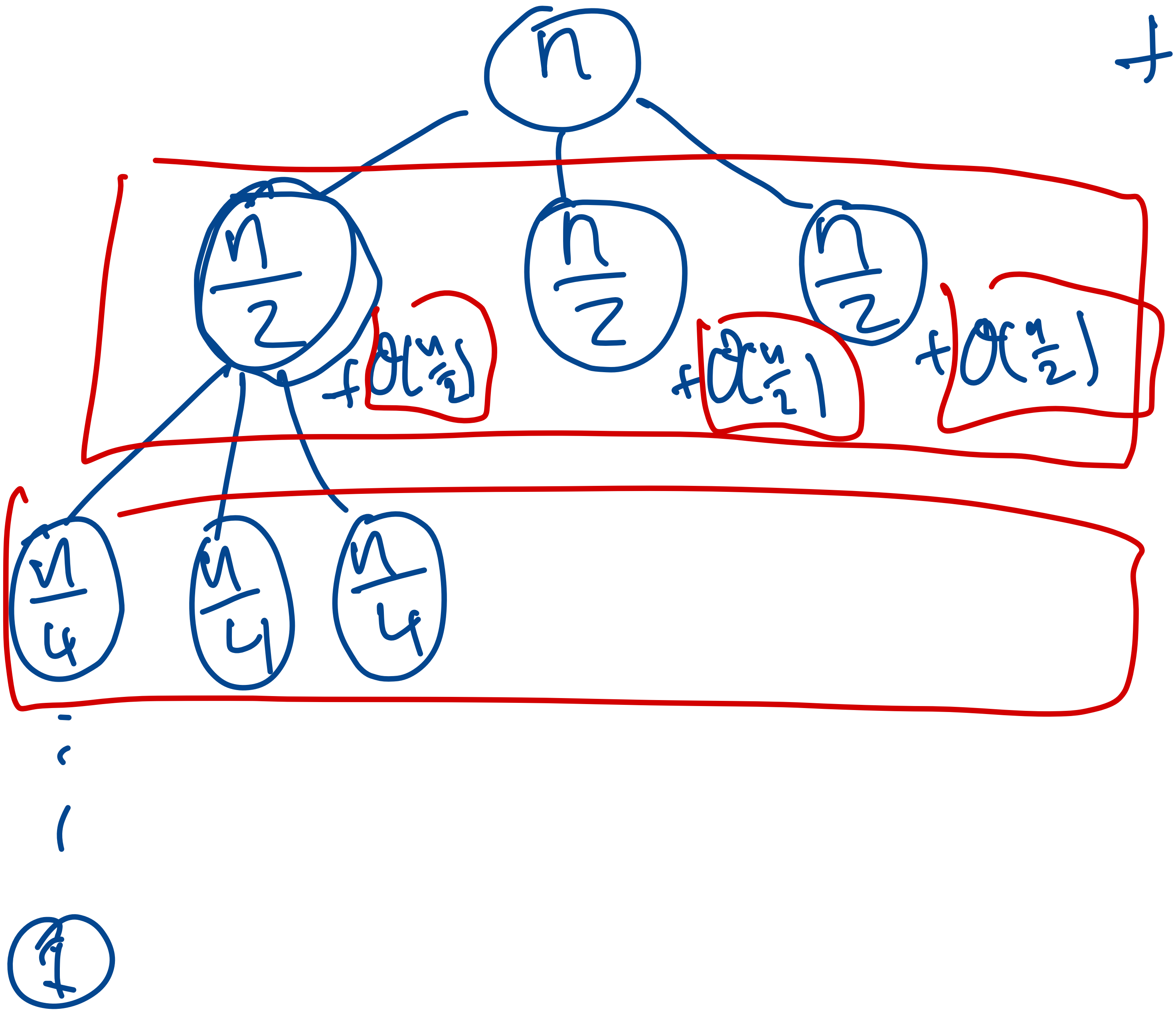
```
KMult(a, b):
  n <- size(a) (= size(b))
  if n = 1 then return a*b
    a = a1 a0
    b = b1 b0
    c2 <- KMult(a1, b1)
    c0 <- KMult(a0, b0)
    c <- KMult(a1 + a0, b1 + b0)
    return (c2 << n) + ((c - c2 - c0) << (n/2)) + c0
```

size $n/2$

$$c_2 \underbrace{00\cdots0}_{n} + c_1 \underbrace{0\cdots0}_{n/2} + c_0$$

Everything but recursive calls runs in $O(n)$ time

Karatsuba Recursion Tree



$$+\ \boxed{\Theta(n)}$$

$$+\left(\frac{3}{2}\right)\Theta(n)$$

$$+\left(\frac{3}{2}\right)^2\Theta(n)$$

$$\vdots$$

$$+\left(\frac{3}{2}\right)^k\Theta(n)$$

$$\vdots$$

$$+\left(\frac{3}{2}\right)^{\log n}\Theta(n)$$

Efficiency of Karatsuba

Tricks:

- $a = 2^{\log a}$
- $(a^b)^c = a^{bc} = (a^c)^b$

At depth k:

- 3^k calls to `KMult`
- size of each call is $n/2^k$
- depth of recursion is $\log n$

Total running time:

- $O(n) + \frac{3}{2}O(n) + \left(\frac{3}{2}\right)^2 O(n) + \cdots + \boxed{\left(\frac{3}{2}\right)^{\log n} O(n)}$

dominates

$\left(\frac{3}{2}\right)^{\log n} O(n) = \frac{3^{\log n}}{2^{\log n}} \cdot O(n) = \frac{3^{\log n}}{n} \cdot O(n) = O(3^{\log n})$

Can show:

- This expression is $O(3^{\log n})$

Simplify:

$3^{\log n} = \left(2^{\log 3}\right)^{\log n} = \left(2^{\log n}\right)^{\log 3} = n^{\log 3} \approx n^{1.58\ldots}$

Final Running Time

Result. The running time of Karatsuba multiplication is $O(n^{\log 3}) \approx O(n^{1.58})$

- when n is reasonbly large, $n^{1.58} \ll n^2$
- E.g., $1,000^2 = 1,000,000$ vs $1,000^{1.58} \approx 55,000$

Recent Progress on Multiplication

Theorem (Harvey and van der Hoeven, 2019). It is possible to multiply two n bit numbers in time $O(n \log n)$.

- See *Mathematicians Discover the Perfect Way to Multiply* from Quanta Magazine
- Main technique: "Fast Fourier Transform," a divide-and-conquer algorithm

Conditional lower bound (Afshani et al., 2019). Multiplying two n bit numbers requires $\Omega(n \log n)$ time, unless the "network coding conjecture" is false.

Recurrence Relation

Running time of Karatsuba multiplication $T(n)$

```
KMult(a, b):
  n <- size(a) (= size(b))
  if n = 1 then return a*b
      a = a1 a0
      b = b1 b0
      c2 <- KMult(a1, b1)
      c0 <- KMult(a0, b0)
      c <- KMult(a1 + a0, b1 + b0)
      return (c2 << n) + ((c - c2 - c0) << (n/2)) + c0
```

all else $O(n)$

3 rec calls of size $\frac{n}{2}$

- Running time satisfies **recurrence relation**

$$T(n) = \underbrace{\boxed{3T(n/2)}}_{\text{rec. calls}} + \underbrace{\boxed{O(n)}}_{\text{rest of time}}$$

- Recurrence of this form satisfies $T(n) = O(n^{\log 3})$

More General Recurrences

- General form of recurrences

\# of recursive calls

size of calls

$$T(n) = \boxed{a}\,T(\boxed{n/b}) + \boxed{f(n)}$$

time to combine sol'ns

- Interpretation for D&C
  - Divide problem into a parts
  - Each part has size n/b
  - Time to combine solutions is $f(n)$

General Solutions

KMult:

- $a = 3$
- $b = 2$
- $c = \log 3$
- $f(n) = \Theta(n) = \Theta(n^1)$

The "Master Theorem"

- $T(n) = aT(n/b) + f(n)$
- Define $c = \log_b a$
- Three cases:

* 1. If $f(n) = O(n^{d})$ for $d < c$ then $T(n) = O(n^c)$ $\quad T(n) = \Theta(n^{\log 3})$

recursion costs more than combining

2. If $f(n) = \Theta(n^c \log^k n)$ then $T(n) = O(n^c \log^{k+1} n)$

recursion comparable to combining

3. If $f(n) = \Omega(n^d)$ for $d > c$, then $T(n) = O(n^d)$

combining more costly than recursive calls

$$f = O(g) \rightsquigarrow f \text{ "}\leq\text{" } g$$

$$f = \Theta(g) \rightsquigarrow f \text{ "}=\text{" } g$$

$$f = \Omega(g) \rightsquigarrow f \text{ "}\geq\text{" } g$$

A Technical Note

In formal statement we have

- $T(n) = aT(n/b) + f(n)$

In practice, might have

- $T(n) = aT(\lceil n/b \rceil) + f(n)$

The conclusion of the theorem still holds in this case!

Master Theorem for Karatsuba

$$T(n) = aT(n/b) + f(n)$$

```
KMult(a, b):
  n <- size(a) (= size(b))
  if n = 1 then return a*b
      a = a1 a0
      b = b1 b0
      c2 <- KMult(a1, b1)
      c0 <- KMult(a0, b0)
      c <- KMult(a1 + a0, b1 + b0)
      return (c2 << n) + ((c - c2 - c0) << (n/2)) + c0
```

Master Theorem for Binary Search

$T(n) = aT(n/b) + f(n)$

```
BinarySearch(a, val, i, j):
  if j = i then return false
  if j - i = 0 then return a[i] = val
  m <- (j + i) / 2
  if a[m-1] >= val then
    return BinarySearch(a, val, i, m)
  else
    return BinarySearch(a, val, m, j)
  endif
```

Master Theorem for MergeSort

$$T(n) = aT(n/b) + f(n)$$

```
MergeSort(a, i, j):
  if j - i = 1 then
    return
  endif
  m <- (i + j) / 2
  MergeSort(a,i,m)
  MergeSort(a,m,j)
  Merge(a,i,m,j)
```

Profit Maximization

Goal. Pick day b to buy and day s to sell to maximize profit.

Formalizing the Problem

Input. Array a of size n

- $a[i] =$ price of Alphabet stock on day i

Output. Indices b (buy) and s (sell) with $1 \le b \le s \le n$ that maximize profit

- $p = a[s] - a[b]$

Simple Procedure

Devise a procedure to determine max profit in time $O(n^2)$.

Divide and Conquer?

Question. Can we compute maximum profit faster?

- Use divide an conquer?